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## Quarks and Anomalies

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A nonperturbative understanding of neutral pion decay was an essential step towards the idea that strong interactions are governed by a color gauge theory for quarks. Some aspects of this work and related problems are still important.

### 1. Quarks before QCD

Any Caltech theory student in the late 1960's, particularly if Murray Gell-Mann was their supervisor, had to be good at distinguishing various "quark models". Were we talking about "constituent" or "current" quarks, and within those categories, was model dependence an issue? Quarks were somehow fundamental, but it was not even clear that their dynamics should be governed by a local field theory. The main tactic was to "abstract" rules which seemed to be model independent and led to physical consequences which could be compared with existing data.

By that time, the quark idea was several years old, dating from work completed independently by the end of 1963: Gell-Mann's quarks,<sup>1</sup> Zweig's aces,<sup>2</sup> and (a reference I have just heard of) Petermann's "spineurs (avec) ... des valeurs non entières de la charge".<sup>3</sup> These papers had in common

- (1) structures  $q\bar{q}$  for mesons and  $qqq$  for baryons built from non-relativistic *constituent* quarks  $q$  and anti-quarks  $\bar{q}$ ,
- (2) the idea that  $SU(3)$  mass formulas<sup>5,6</sup> are due to the strange quark  $s$  being heavier than the up and down quarks  $u, d$ , and
- (3) concerns about whether the fractional charges would be observable.

Gell-Mann and Zweig were led to (1) by the need to explain the absence

of exotic  $SU(3)$  multiplets in the Eightfold Way.<sup>5,7</sup> Zweig analyzed the constituent quark model in detail, deriving properties such as spins, parities and masses for various  $SU(3)$  multiplets. Gell-Mann had a separate aim: to reproduce current algebra, a set of equal-time commutators for  $SU(3) \times SU(3)$  currents<sup>4</sup> which he had previously managed to abstract without using quarks. For this, he needed *current quarks*, i.e. relativistic fields  $q(x)$  and  $\bar{q}(x)$  for each flavor  $q = u, d, s$ , from which electromagnetic, weak and other  $SU(3) \times SU(3)$  currents could be constructed.

Immediately, there were concerns about constituent quark statistics. How can a baryon like  $\Sigma^{++}$  exist as an  $S$ -wave spin-flavor symmetric state  $|u\uparrow u\uparrow u\uparrow\rangle$  if quarks are spin- $\frac{1}{2}$  fermions? It is hard to imagine ground states being  $P$ -wave, so instead, it was proposed that quarks are either<sup>9</sup> parafermions<sup>10</sup> of order 3 or<sup>11–13</sup> fermions with an extra quantum number taking three values, which we now know as *color*.<sup>14–16</sup> The observed fermionic baryons  $|qqq\rangle$  are then symmetric in space-spin-flavor (a) for paraquarks automatically, or (b) for fermion quarks antisymmetrized in a color  $SU(3)$  singlet state<sup>12,13</sup> (but not  $SO(3)$ , because that would allow colorless diquark states  $|qq\rangle$ ).

Whether para-particle or colored multiplets would appear at higher energies or be banned completely (quark confinement) was not clear. In an attempt to make these extra states appear less weird, colored quarks were initially given integer charges<sup>12,13</sup> which, however, depended on the color index. Then photons could excite color from hadrons and perhaps induce transitions to a deconfined (weird) sector.

In the model eventually adopted in 1972,<sup>15,16</sup> quarks became colored fermions with fractional charges, with 3 colors for each charge or flavor. As a result, the electromagnetic and weak currents became color  $SU(3)$  singlets, like the observed hadronic spectrum. Confinement was as unclear for this model as the others. If confinement were not absolute, the model could have degenerate color multiplets and fractionally charged states above some threshold energy. Comparing all of these models, it was concluded that, as models of *constituent* quarks, they were hard to distinguish below thresholds for deconfinement.

However the 1972 model was also designed to take into account color for *current* quarks. The rest of this article describes how studies of short-distance behavior<sup>17</sup> and the reaction<sup>18</sup>  $\pi^0 \rightarrow \gamma\gamma$  led to this.

## 2. Scale Invariance at High Energies

I started life at Caltech as a graduate student in the fall of 1968. The very first seminar, on Tuesday October 1, was “Partons” by Richard P. Feynman, with Murray Gell-Mann sitting near the front. Feynman had just returned from a summer in SLAC hearing about Bjorken’s work<sup>19</sup> on scaling in deep inelastic lepton-nucleon scattering and developing a model of point scatterers (partons) to give the same results. Murray kept asking “but Richard, what are their quantum numbers? Are they quarks?” but Richard’s sole concern was scaling due to scattering by “grains of sand inside the nucleon”. (A year or so later, my fellow student Finn Ravndal got him interested in quarks.)

Murray began supervising me two months later and in due course asked me, as an initial research exercise, to try using the Cutkosky bootstrap model<sup>20</sup> to generate higher symmetries like  $SU(6)$ . That produced hundreds of equations. Fortunately, just a few of them could be used to show that there could be no consistent solution. Murray commented that he hadn’t intended the exercise “to be so vigorous” and suggested that I take a trip while he thought of a suitable PhD topic. My fellow student Chris Hamer and I had already planned to drive around the US that summer (1969), so we left immediately and on the way back, stopped at Aspen.

Murray had just started working on scale invariance as an approximate symmetry of hadrons, and suggested that I do the same. This would involve the energy-momentum tensor  $\theta_{\mu\nu}$  as well as the  $SU(3) \times SU(3)$  currents. Did I know about the Belinfante<sup>21</sup> tensor? Fortunately, I did (from Geoff Opat, supervisor of Chris and myself as Masters students in Melbourne, 1966-68). In that case, the next step was to understand all 14 pages of Wilson’s paper on operator product expansions.<sup>17</sup>

Wilson generalised current algebra, replacing equal-time limits of commutators by short-distance limits of products of currents and other observables such as  $\theta_{\mu\nu}$ . Instead of a single term on the right-hand side, he obtained an asymptotic expansion  $\sum_n \mathcal{C}_n O_n$  with coefficient functions

$$\mathcal{C}_1 \gg \mathcal{C}_2 \gg \mathcal{C}_3 \gg \dots \quad (1)$$

in order of decreasing singularity times observable operators  $O_1, O_2, O_3 \dots$  of increasing operator dimensionality (in mass units). Equal-time commutators, such as in Gell-Mann’s current algebra and Bjorken’s work on scaling, could be recovered by noting that, since commutators vanish for space-like separations, their equal-time limits are controlled by the short-distance behavior of the relevant operator product. Checks in renormalized perturbation theories or for free current quarks indicated that, apart from quantum

number constraints, the same set of operators  $\{O_n\}$  tended to appear in the expansion, whatever the operator product used to generate them: “a limited set of licensed operators”, as Murray put it.

A key feature of Wilson’s work was his critique<sup>22</sup> of canonical field theory: operators usually *cannot* be multiplied at the same point, equal-time commutators may be singular, and  $T$ -ordering with step functions  $\theta(t - t')$  can fail. These *anomalies* arise wherever renormalization is necessary. In particular, renormalized perturbation theory produces  $\log^p(\mu^2(x - y)^2)$  factors at short distances, where  $\mu$  is the renormalization scale. When summed up à la Gell-Mann and Low,<sup>23</sup> anomalous powers may be produced. If the ultraviolet limit is controlled by a nontrivial Gell-Mann–Low fixed point, scale invariance becomes exact at short distances, with anomalous dimensions for all operators  $O_n$  except those which are conserved or partially conserved. I was happy to abstract these rules and learn the renormalization group later.

Wilson’s paper<sup>17</sup> also featured a Sec. VII “Applications” with five subsections, each equivalent to a separate publication. Subsection D “ $\pi^0 \rightarrow \gamma\gamma$  Problem” drew my attention because (a) it explained how short-distance singularities determine contact terms in low-energy Ward identities and (b) I had seen the papers of Bell and Jackiw<sup>24</sup> and Adler<sup>25</sup> on the axial anomaly. Could the three-point function  $T\langle \text{vac} | J_\alpha J_\beta J_{\mu 5} | \text{vac} \rangle$  of the electromagnetic and axial-vector currents  $J_\alpha$  and  $J_{\mu 5}$  be determined at short distances without using perturbation theory? Noting Wilson’s comment (Sec. VIII) that “the prospects for obtaining such a solution seem dim at present”, I filed the problem away as a challenge for the future.

At that time, the main question was whether Bjorken scaling is exact or not. Bjorken<sup>19</sup> obtained scaling by assuming that an infinite set of equal-time commutators of  $J_\alpha$  with its derivatives is finite, i.e. not zero. It was quickly established that this was equivalent to assuming canonical or free-field (parton) behavior for the coefficient functions (1). Towers of these short-distance singularities could be summed to form terms in an operator product expansion near the light cone  $(x - y)^2 \rightarrow 0$ , the limit in position space conjugate to Bjorken’s limit.<sup>26,27</sup> By then, quarks were widely believed to be responsible for scaling, so the proposal of Fritzsche and Gell-Mann<sup>28</sup> to abstract the light-cone expansion from free-quark theory was logical.

The argument against exact Bjorken scaling was led by Wilson.<sup>29</sup> Interactions tend to increase the dimensions of composite-field operators  $O_n$  which are not conserved exactly or partially, making higher- $n$  functions (1) less singular on the light cone. The difficulty for this point of view was explaining why these anomalous corrections were *all* so small. Neverthe-

less, I tended to belong to this school of thought. My concern was that any tensor operator  $O(x)$  lacking an anomalous dimension would be at least partially conserved, because the leading singularity of  $\langle \text{vac} | O(x) O(y) | \text{vac} \rangle$  would be canonical and hence divergenceless. Therefore, my view was that the *only* operators allowed to have canonical dimension were  $\theta_{\mu\nu}$  and the  $SU(3) \times SU(3)$  currents.

In particular, there was the  $U(1)$  problem, which I knew from Gell-Mann's 1969 Hawaii lectures.<sup>30</sup> If we abstract from the free-quark model, the isoscalar current

$$J_{\mu 5}^0 = \bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d \quad (2)$$

is conserved in the  $SU(2) \times SU(2)$  limit. This is a disaster<sup>31</sup> because then the  $SU(2) \times SU(2)$  condensate

$$\langle \text{vac} | \bar{u}u + \bar{d}d | \text{vac} \rangle \neq 0 \quad (3)$$

*also* acts as an axial  $U(1)$  condensate. In addition to  $\pi^+, \pi^0, \pi^-$ , there would have to be a fourth Nambu-Goldstone boson, an isoscalar  $0^-$  meson of mass  $O(m_\pi)$ . If just one extra conserved current could cause so much trouble, we certainly did not want an infinite tower of them.

The choice between canonical and anomalous dimensions would be cleared up by asymptotic freedom<sup>32,33</sup> two years later. Only  $\theta_{\mu\nu}$  and the  $SU(N_f) \times SU(N_f)$  currents behave canonically. Coefficients  $\mathcal{C}_n$  of other operators  $O_n$  have their canonical behavior modified by inverse logarithmic powers, corresponding to a very weak violation of Bjorken scaling. The  $U(1)$  problem was not so easily dismissed, and while majority opinion is that it is understood, they all miss the reference<sup>34</sup> where problems yet to be resolved are analysed.

### 3. Approximate Scale Invariance at Low Energies

At the same time (1969-71), I was supposed to be working on my PhD research project. Clearly the hadronic ground state  $|\text{vac}\rangle$  breaks scale invariance very strongly, given the 1 GeV scale set by baryons. This could be simply due to scale invariance being badly broken explicitly, with the trace  $\theta_\mu^\mu$  large as an operator. The alternative is that scale invariance is approximately conserved in the Nambu-Goldstone mode with a massless  $0^{++}$  *dilaton* in the limit  $\theta_\mu^\mu \rightarrow 0$ .

The analogy with chiral symmetry was obvious. Both chiral  $SU(3) \times SU(3)$  and scale symmetry would be manifest at short distances and hidden elsewhere by the effects of their Goldstone bosons, the  $0^-$  octet  $\pi, K, \eta$  and

the  $0^+$  singlet dilaton  $\sigma$  (not to be confused with the field “ $\sigma$ ” of the sigma model). I chose the simplest case where scale invariance was the result of taking the chiral  $SU(3) \times SU(3)$  limit, so the 3-flavor version of the chiral condensate (3) could also act as a scale condensate.

This picture is no longer entirely accurate, given that QCD renormalization effects break scale symmetry everywhere, including short distances. However if 3-flavor QCD has an infrared fixed point  $\alpha_{\text{IR}}$ , where  $\theta_\mu^\mu$  vanishes apart from  $O(m_{u,d,s})$  corrections, the essential features of the original scheme can be reproduced by a double expansion in the running gluon coupling  $\alpha_s$  about  $\alpha_{\text{IR}}$  and the light quark masses  $O(m_{u,d,s})$  about zero.<sup>35</sup>

The dilaton idea is contained in footnote 38 of the 1962 current algebra paper.<sup>4</sup> A “resonance or quasi-resonance” which dominates a dispersion relation for

$$\langle \text{particle} | \theta_\mu^\mu | \text{particle} \rangle = \text{particle mass} \quad (4)$$

yields “a relation of the Goldberger-Treiman type” where “the coupling of the resonant state to different particles is roughly proportional to their masses”. In the scale-invariant limit,<sup>30</sup> the vacuum would become degenerate, as for exact chiral symmetry, except for the degeneracy being non-compact. Physical predictions are then the result of expanding in  $m_\sigma^2$  about zero.

The term “dilaton” is often used in a manner which is distinct from the scheme above or even contradicts it. The earliest variant was Fujii’s proposal<sup>37</sup> of a finite-range scalar component of gravity. Gell-Mann called it a “Brans-Dickeon” after the well-known proponents of the scalar-tensor theory of gravity,<sup>38</sup> but the name did not stick. In modern times, “dilaton” is often used for a scalar particle which has zero mass classically but becomes massive due to quantum corrections, such as Higgs bosons which acquire mass due to dimensional transmutation.<sup>39</sup> Since there is no way of “turning off” such a mass, this has nothing to do with dilatons in the original sense.

In my student days, the main candidate for  $\sigma$  was  $\epsilon(700)$ , whose existence was not clear. Final state pions interact very strongly in the  $0^{++}$  channel, so there was good reason to assume the presence of a resonance far off shell. However that meant that it was very hard to pin down in phase-shift analyses. It was declared dead in the 1976 particle data tables, but in recent years, has been resurrected as the broad but clearly defined resonance<sup>36</sup>  $f_0(500)$ .

If dilatons couple to mass, why is its coupling to pions so large? In leading order, one would expect  $F_\sigma g_{\sigma\pi\pi}$  to be  $2m_\pi^2$  for the coupling  $g_{\sigma\pi\pi} \sigma \cdot \pi \cdot \pi$ , where

$F_\sigma$  is the analogue of the pion decay constant  $F_\pi \simeq 93$  MeV and has a similar order of magnitude:

$$\langle \sigma(q) | \theta_{\mu\nu} | \text{vac} \rangle = (F_\sigma/3)(q_\mu q_\nu - g_{\mu\nu} q^2). \quad (5)$$

The solution, on which I based my PhD thesis, was to note that the result is really

$$F_\sigma g_{\sigma\pi\pi} = 2m_\pi^2 + O(m_\sigma^2) \quad (6)$$

and use approximate chiral symmetry to deduce the coefficient of  $m_\sigma^2$ :

$$F_\sigma g_{\sigma\pi\pi} = -m_\sigma^2 + O(m_\pi^2). \quad (7)$$

This implies a width of a few hundred MeV, as required. I did it the hard way, using basic current algebra, and so took too long to obtain a mass formula for  $m_\sigma^2 F_\sigma^2$ . In the meantime, John Ellis was working on his PhD in Cambridge (UK), and obtained both Eq. (7) and the mass formula by efficient use of a chiral-scale effective Lagrangian. A few months later, we met and were able to compare notes at the 1971 Coral Gables conference.<sup>40,41</sup>

No account of these times would be complete without mentioning the episode in 1970 when Feynman became excited about Bose statistics for quarks. He hoped to explain the  $\Delta I = 1/2$  rule for nonleptonic decays of strange particles. How quarks could possibly be bosons was a matter for future study; perhaps their bad statistics would not matter if they were confined. Almost immediately, we heard that the idea had already been suggested,<sup>43,44</sup> but the interest generated by Feynman<sup>45</sup> in this key problem was good for particle physics. A few months later, the correct version of the idea was proposed<sup>46</sup> (also anticipated in Japan<sup>47</sup>): for fermion quarks with color (and even for paraquarks<sup>48</sup>), the color antisymmetrization of  $qqq$  states plus current algebra implies the  $\Delta I = 1/2$  rule for nonleptonic hyperon decays, but says nothing about  $\Delta I = 1/2$  for  $K \rightarrow \pi\pi$ .

Since nonleptonic strange particle decays had been a problem for so long,<sup>42</sup> my interest was piqued. I told Murray of this, carefully avoiding any suggestion that quarks could be bosons (which I didn't believe anyway), and drew the response "watch out, it's a can of worms!" I was too busy finishing my PhD to pursue it; otherwise, I may have drawn Fig. 1, which is required by approximate chiral-scale invariance. It shows that the  $\Delta I = 1/2$  rule for kaons is due to a large contribution from the dilaton pole. Only after 40-odd years, with help from my young colleague Lewis Tunstall, can I report a solution to that problem.<sup>35</sup> For hyperon decays, the  $\Delta I = 1/2$  rule is understood, but current algebra does not seem to work: that part of the problem is still a can of worms.

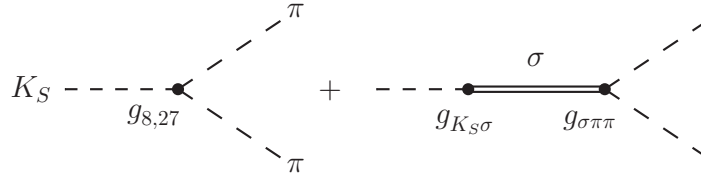


Fig. 1. Tree diagrams in chiral-scale perturbation theory<sup>35</sup> for  $K_S \rightarrow \pi\pi$ . The vertex amplitudes due to **8** and **27** contact couplings  $g_8$  and  $g_{27}$  are dominated by the  $\sigma/f_0$  pole amplitude. The magnitude of  $g_{K_S\sigma}$  can be deduced from  $K_S \rightarrow \gamma\gamma$  and  $\gamma\gamma \rightarrow \pi\pi$ .

After Coral Gables, there was a thesis to be written, and a suitable way of ending it had to be found. What else could dilatons do?

From the literature on axial anomalies, I knew about Schwinger's 1951 paper on gauge invariance.<sup>49</sup> In Sec. V, he obtained unique results for both  $\pi^0 \rightarrow \gamma\gamma$  and  $\sigma \rightarrow \gamma\gamma$  in one-loop Yukawa theory by imposing gauge invariance on the renormalization procedure. In terms of the electromagnetic field tensor  $F_{\mu\nu}$ , fermion mass  $M$ , Yukawa coupling  $g$ , and fine-structure constant  $\alpha$ , the answer for  $\sigma \rightarrow \gamma\gamma$  is

$$\mathcal{L}_{\sigma\gamma\gamma}^{\text{Yukawa}} = -\frac{\alpha g}{6\pi M} \sigma F^{\mu\nu} F_{\mu\nu} \quad (8)$$

In the second-last paragraph of my thesis, I noted that this breaks scale invariance (operator dimension  $\neq 4$ ), so if  $M$  plays the role of  $F_\sigma$  as well as  $F_\pi$ , perhaps both  $F_\pi g_{\pi\gamma\gamma}$  and  $F_\sigma g_{\sigma\gamma\gamma}$  are anomalous. Already, Wilson had shown<sup>22</sup> that one-loop corrections in  $\lambda\phi^4$  theory break scale invariance, which he interpreted as an anomaly in the trace of  $\theta_{\mu\nu}$ . Perhaps there is an electromagnetic trace anomaly due to strong interactions? I was moving to a post-doctoral job at Cornell; as soon as I arrived, I would try to extend Wilson's method for  $\pi^0 \rightarrow \gamma\gamma$  to  $\sigma \rightarrow \gamma\gamma$ .

#### 4. Derivation of $\pi^0 \rightarrow \gamma\gamma$ for Nonperturbative Pions

When Schwinger analysed  $\pi^0 \rightarrow \gamma\gamma$ , chiral invariance and PCAC (partially conserved axial current) were unknown. At issue was the equivalence

$$\phi \bar{\psi} \gamma_5 \psi \leftrightarrow -(i/2M) \bar{\psi} \gamma_\mu \gamma_5 \psi \partial^\mu \phi, \quad \phi = \pi^0 \text{ field} \quad (9)$$

between pseudoscalar and pseudovector couplings for the one-fermion-loop triangle diagram. The trouble was that the product of the fermion fields at the same point is singular. The solution was to consider  $\psi$  and  $\bar{\psi}$  at different points  $x'$  and  $x''$  and make the analysis gauge invariant: then the limit  $x' \rightarrow x''$  becomes finite. Rephrased in terms of chiral symmetry, the



problem is that the Noether construction fails because (a) it requires

$$\frac{\partial \mathcal{L}}{\partial \partial^\mu \psi} = \bar{\psi} \gamma_\mu \quad \text{and} \quad \delta_{\text{axial}} \psi = \gamma_5 \psi \quad (10)$$

to be multiplied at the same point and (b) it does not work for non-local expressions produced by point splitting. The axial anomaly<sup>24,25</sup> is responsible for this failure: it is the finite counterterm mismatch between gauge invariant and chiral invariant renormalization prescriptions for axial-vector operators.

Wilson's version of this was designed to avoid perturbation theory. In particular, if pions are  $q\bar{q}$  states which become Nambu-Goldstone bosons in the chiral limit, they are certainly not perturbative and so should not be represented by a perturbative field  $\phi$ .

The other key feature of his approach was the use of short distance analysis. The connection between axial and trace anomalies and short distance behavior is best illustrated by considering first how equal-time commutators produce contact terms  $\sim \delta^4(x - y)$  in ordinary Ward identities.

Given a free massive boson field  $\varphi$ , let  $\partial_\mu \varphi$  play the role of a current. Canonically, the divergence of  $T\{\varphi \partial_\mu \varphi\}$  is found by writing the  $T$ -product in terms of step functions  $\theta(\pm x_0)$  and unordered field products, differentiating the step functions

$$\frac{\partial}{\partial x^\mu} \theta(\pm x_0) = \pm \delta(x_0) g_{0\mu} \quad (11)$$

and substituting  $\partial^2 \varphi = -m^2 \varphi$ :

$$\partial^\mu T\{\varphi(0) \partial_\mu \varphi(x)\} = [\partial_0 \varphi(x), \varphi(0)] \delta(x_0) - m^2 T\{\varphi(0) \varphi(x)\}. \quad (12)$$

In this case, the contact term can be found by substituting a canonical commutator:

$$[\partial_0 \varphi(x), \varphi(0)] \delta(x_0) = -i \delta^4(x) I, \quad I = \text{identity operator}. \quad (13)$$

The short-distance method is to note that a term  $\sim \delta^4(x)$  can arise only if  $\partial^\mu$  acts on a singularity  $\sim 1/x^3$  at  $x \sim 0$ . The leading term of the operator product expansion for  $T\{\varphi(0) \partial_\mu \varphi(x)\}$  is given by the propagator of the massless theory

$$T\{\varphi(0) \partial_\mu \varphi(x)\} \rightarrow \frac{x_\mu}{2\pi^2(x^2 - i\epsilon)^2} I, \quad I = \text{identity operator} \quad (14)$$

Substituting  $\partial^\mu (x_\mu/x^4) = -2i\pi^2 \delta^4(x)$  and  $\partial^2 \varphi = -m^2 \varphi$ , we find

$$\partial^\mu T\{\varphi(0) \partial_\mu \varphi(x)\} = -i \delta^4(x) I - m^2 T\{\varphi(0) \varphi(x)\}. \quad (15)$$

in agreement with Eqs. (12) and (13).

For the axial anomaly, the problem is to evaluate the quantity

$$S = -\frac{\pi^2}{12} \epsilon^{\mu\nu\alpha\beta} \iint d^4x d^4y x_\mu y_\nu T \langle \text{vac} | J_\alpha(x) J_\beta(0) \partial^\gamma J_{\gamma 5}(y) | \text{vac} \rangle, \quad (16)$$

where data for  $\pi^0 \rightarrow \gamma\gamma$  and approximate  $SU(2) \times SU(2)$  symmetry imply  $S \simeq +0.5$ . The constant  $S$  normalizes the contact term in an anomalous Ward identity of the form

$$\begin{aligned} & \partial_y^\nu \langle \text{vac} | J_\alpha(x) J_\beta(0) J_{\nu 5}(y) | \text{vac} \rangle \\ &= \frac{S}{2\pi^2} \epsilon_{\alpha\beta\mu\nu} \partial_x^\mu \partial_y^\nu \delta^4(x) \delta^4(y) + T \langle \text{vac} | J_\alpha(x) J_\beta(0) \partial^\nu J_{\nu 5}(y) | \text{vac} \rangle. \end{aligned} \quad (17)$$

The contact term scales as  $1/\{\text{length}\}^{10}$ , so it must be generated by a short distance singularity

$$J_\alpha(x) J_\beta(0) J_{\nu 5}(y) \sim 1/\{\text{length}\}^9 \quad (18)$$

as *both*  $x_\mu$  and  $y_\mu$  tend to zero. (Do not confuse this with the short-distance properties of  $J_\alpha J_\beta \partial^\gamma J_{\gamma 5}$  in Eq. (16), where the condition  $\dim \partial^\gamma J_{\gamma 5} < 4$  ensures convergence of the integral.)

In Eq. (17), a *single* derivative  $\partial_y^\mu$  produces a *product of two delta functions*, so it is clear that  $\theta$ -functions in time *cannot* be used to construct “ $T$ ”. This example exposes the limitations of canonical field theory very effectively.

In perturbation theory, it has long been known<sup>50</sup> but not often noted that time ordering is part of the renormalization procedure. In general, “ $T$ ” must be regarded as an operation which depends on the renormalization prescription. The difference between two time-ordering procedures for a given operator product is a set of contact terms at coinciding points. In the case of the triangle diagram coupled to photons, electromagnetic gauge invariance specifies the renormalization procedure completely:

$$“T” \rightarrow T_{e'_{\text{mag}}}. \quad (19)$$

Wilson<sup>17</sup> circumvented the “ $T$ ” problem by excising a small neighbourhood around lines of coinciding points in the integral (16). Let the region of integration be restricted to the region

$$\mathcal{R} = \{ |x_0| > \epsilon, |y_0| > \epsilon', |x_0 - y_0| > \epsilon'' \}. \quad (20)$$

shown in Fig. 2, so that Eq. (16) becomes

$$S = -\frac{\pi^2}{12} \epsilon^{\mu\nu\alpha\beta} \iint_{\mathcal{R}} d^4x d^4y x_\mu y_\nu T \langle \text{vac} | J_\alpha(x) J_\beta(0) \partial^\gamma J_{\gamma 5}(y) | \text{vac} \rangle + O(\epsilon, \epsilon', \epsilon''). \quad (21)$$

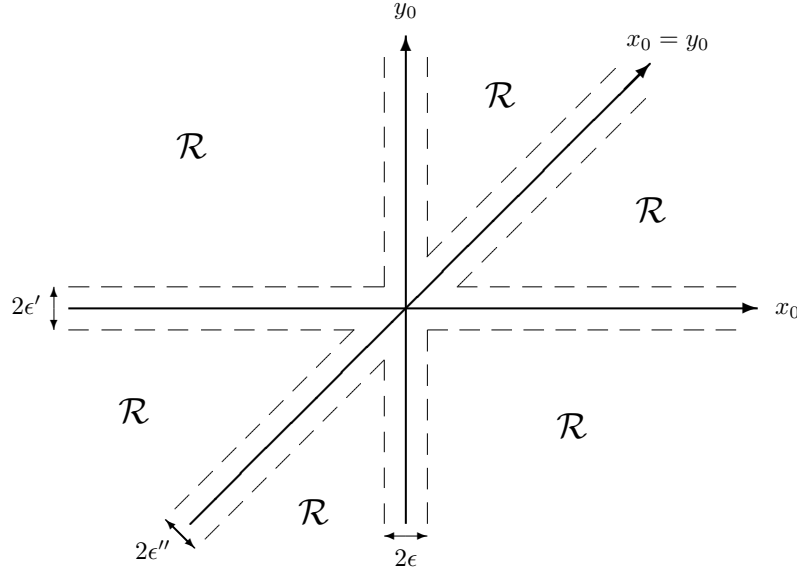


Fig. 2. Integration region  $\mathcal{R}$  defined by Eq. (20).

Of course,  $S$  does not depend on  $\epsilon$ ,  $\epsilon'$ , or  $\epsilon''$ . Within  $\mathcal{R}$ , define

$$\begin{aligned} X_\gamma &= \epsilon^{\mu\nu\alpha\beta} x_\mu y_\nu T \langle \text{vac} | J_\alpha(x) J_\gamma(0) J_{\beta 5}(y) + J_\gamma(x) J_\alpha(0) J_{\beta 5}(y) | \text{vac} \rangle, \\ Y_\gamma &= \epsilon^{\mu\nu\alpha\beta} x_\mu y_\nu T \langle \text{vac} | J_\alpha(x) J_\beta(0) J_{\gamma 5}(y) + J_\alpha(x) J_\gamma(0) J_{\beta 5}(y) | \text{vac} \rangle \end{aligned} \quad (22)$$

where now time ordering with  $\theta$ -functions is allowed because  $\mathcal{R}$  excludes coinciding points. This also means that derivatives commute with the  $T$ -operation, so we can obtain

$$S = -\frac{\pi^2}{12} \iint_{\mathcal{R}} d^4x d^4y (\partial_x^\gamma X_\gamma + \partial_y^\gamma Y_\gamma) + O(\epsilon, \epsilon', \epsilon'') \quad (23)$$

by using current conservation  $\partial^\gamma J_\gamma = 0$ , translation invariance of  $|\text{vac}\rangle$  and symmetry  $x \leftrightarrow y$  of the integral to  $O(\epsilon, \epsilon', \epsilon'')$ . If  $\Sigma$  is the surface in 8-dimensional space which bounds  $\mathcal{R}$ , we have

$$S = -\frac{\pi^2}{12} \int_{\Sigma} d\vec{\Sigma} \cdot \vec{Z} + O(\epsilon, \epsilon', \epsilon'') \quad (24)$$

where  $\vec{Z} = (X_\gamma, Y_\gamma)$  is an 8-dimensional vector formed from the components of  $X_\gamma$  and  $Y_\gamma$ .

So  $S$  is given by the result of taking  $\epsilon$ ,  $\epsilon'$  and  $\epsilon''$  to zero in Eq. (24). Since the current operators commute at space-like separations, their products at short distances are all that we need. If we consider (say)  $\epsilon \rightarrow 0$  and exclude

the  $x, y \sim 0$  neighbourhood where the axes in Fig. 2 meet, we have  $x \sim 0$  for fixed  $y$ , which means expanding in  $J_\alpha(x)J_\beta(0)$  to produce an equal-time commutator. There could be three commutators in principle, one for each axis, but explicit checks confirm the conclusion<sup>51</sup> that they all vanish. Therefore  $S$  is entirely determined by the leading VVA short-distance singularity

$$T\{J_\alpha(x)J_\beta(0)J_{\gamma 5}(y)\} \sim G_{\alpha\beta\gamma}(x, y)I, \quad x, y \sim 0, \quad (25)$$

so it can be calculated if the three-point function  $G_{\alpha\beta\gamma}$  is known.

At this point, I tried the same analysis for the trace anomaly. Let the amplitude for photons to couple to the hadronic energy-momentum tensor be

$$\langle \gamma(\epsilon_1, k_1) \gamma(\epsilon_2, k_2) | \theta_\mu^\mu(0) | \text{vac} \rangle = (\epsilon_1 \cdot \epsilon_2 k_1 \cdot k_2 - \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1) F((k_1 + k_2)^2). \quad (26)$$

As in Eq. (16) for  $S$ , the trace anomaly corresponds to the low-energy limit  $k_1, k_2 \sim 0$ :

$$F(0) = -\frac{\pi\alpha}{3} \iint d^4x d^4y x \cdot y \langle \text{vac} | J^\alpha(x) J_\alpha(0) \theta_\mu^\mu(y) | \text{vac} \rangle. \quad (27)$$

The aim was to substitute the formula for the divergence of the conformal current

$$\partial_y^\mu \{ (2y_\lambda y^\nu - \delta_\lambda^\nu y^2) \theta_{\mu\nu}(y) \} = 2y_\lambda \theta_\mu^\mu(y) \quad (28)$$

and integrate by parts. To my surprise, I found that it was not necessary to exclude coinciding points as in Fig. 2. Instead, I found that an answer could be found directly by restricting just the  $x$  integration to  $|x_0| > \eta$  for small  $\eta > 0$  to keep the  $x, y \sim 0$  singularity

$$T\{J_\alpha(x)J_\beta(0)\theta_{\mu\nu}(y)\} \sim K_{\alpha\beta\mu\nu}(x, y)I \quad (29)$$

under control. Then integration by parts with respect to  $y$  produced known equal-time commutators, so the  $y$  integral could be done, with the result

$$F(0) = -\frac{i\pi\alpha}{6} \int_{|x_0|>\eta} d^4x \partial_x^\nu \{ x^2 x_\nu T \langle \text{vac} | J^\alpha(x) J_\alpha(0) | \text{vac} \rangle \} + O(\eta). \quad (30)$$

We have  $J_\alpha J_\beta \sim R/x^6$  at short distances, where  $R$  is the asymptotic Drell-Yan ratio

$$R = \{ (\sigma(e^+e^- \rightarrow \text{hadrons}) / (\sigma(e^+e^- \rightarrow \mu^+\mu^-)) \}_{\text{energy} \rightarrow \infty} \quad (31)$$

so the  $x$  integral can also be done, yielding an exact result:

$$F(0) = 2R\alpha/3\pi. \quad (32)$$

In effect, an anomalous term<sup>a</sup>  $(R\alpha/6\pi)F_{\mu\nu}F^{\mu\nu}$  is induced in the trace of the energy-momentum tensor by electromagnetism.<sup>18</sup> The same result was found independently by Mike Chanowitz and John Ellis<sup>52</sup> via a momentum-space analysis. It was the immediate precursor of the gluonic trace anomaly  $\beta(\alpha_s)/(4\alpha_s)G_{\mu\nu}^a G^{a\mu\nu}$  found a few years later.<sup>53</sup>

The unexpected feature of the analysis leading to Eq. (32) was that, although the 3-point singularity  $K_{\alpha\beta\mu\nu}$  is responsible for the presence of the trace anomaly, its full functional form is not needed: only the subregion  $x - y \ll x, y$  within the  $x, y \sim 0$  region is needed. In Fig. 2, this subregion connects the central area  $x, y \sim 0$  to other  $x \sim y$  regions along the diagonal axis  $x_0 = y_0$ .

That led me to consider *nested* operator product expansions, where an expansion such as

$$T\{A(x)B(0)\} \sim \sum_m \mathcal{C}_m(x)O'_m(0) \quad \text{for } x \sim 0 \quad (33)$$

is substituted into a larger expansion, e.g.

$$T\{A(x)B(0)C(y)\} \sim \sum_n f_n(x, y)O_n(0) \quad \text{for } x, y \sim 0. \quad (34)$$

This is legitimate provided that  $y$  is independent of the limit  $x \rightarrow 0$ , i.e.  $x \ll y$ . Then a subsequent limit  $y \rightarrow 0$  can be taken:

$$T\{O'_m(0)C(y)\} \sim \sum_n \mathcal{C}_{mn}(y)O_n(0). \quad (35)$$

The result is a set of consistency conditions<sup>18</sup>

$$f_n(x, y) \sim \sum_m \mathcal{C}_m(x)\mathcal{C}_{mn}(y). \quad (36)$$

The idea works at short distances (and not on other parts of light cones) provided the limits are *nested*. For the example above, fix  $\hat{x}$  and  $\hat{y}$  in

$$x = \rho_1 \rho_2 \hat{x} \quad \text{and} \quad y = \rho_2 \hat{y} \quad (37)$$

and take the limits  $\rho_1 \rightarrow 0$  and  $\rho_2 \rightarrow 0$  independently. This is the position-space version of Weinberg's limiting procedure<sup>54</sup> used to classify the asymptotic behavior of amplitudes and hence justify power counting methods for renormalization.

An obvious next step was to apply this procedure to the short-distance VVA function  $G_{\alpha\beta\gamma}$  of Eq. (25). Let

$$u = x^2 - i\epsilon, \quad v = y^2 - i\epsilon, \quad w = (x - y)^2 - i\epsilon. \quad (38)$$

<sup>a</sup>The extrapolation in  $(k_1 + k_2)^2$  from zero to  $m_\sigma^2$  used to estimate  $F_\sigma g_{\sigma\gamma\gamma}$  from  $F(0)$  has had to be modified,<sup>35</sup> because  $\pi, K$  loop diagrams compete with the  $\sigma$ -pole amplitude.

The relevant two-point expansions are

$$\begin{aligned} T\{J_\alpha(x)J_\beta(0)\} &\sim R(g_{\alpha\beta}x^2 - 2x_\alpha x_\beta)I/(\pi u)^4 + K\epsilon_{\alpha\beta\lambda\mu}x^\lambda J_5^\mu(0)/(3\pi^2 u^2) \\ T\{J_5^\mu(0)J_{\gamma 5}(y)\} &\sim R'(\delta_\gamma^\mu y^2 - 2y^\mu y_\gamma)I/(\pi v)^4 \end{aligned} \quad (39)$$

where  $R'$  is the isovector part of  $R$ , and  $K$  is measurable in polarised deep-inelastic electroproduction or in  $e^+ + e^- \rightarrow \mu^+ + \mu^- + \pi^0$ . The result

$$G_{\alpha\beta\gamma}(x, y) \xrightarrow{x \ll y} \{K\epsilon_{\alpha\beta\lambda\mu}x^\lambda J_5^\mu(0)/(3\pi^2 u^2)\} R'(\delta_\gamma^\mu y^2 - 2y^\mu y_\gamma)I/(\pi v)^4. \quad (40)$$

said something about the normalization of  $G_{\alpha\beta\gamma}$ , but without a formula valid for the whole  $x, y \sim 0$  region, the calculation of  $S$  could not be completed.

In the meantime, I was checking products of  $\theta_{\mu\nu}$  with other currents to see if the absence of the soft trace at short distances would imply asymptotic<sup>b</sup> conformal invariance, as indicated by Eq. (28). Satisfied that it did, I required conformal invariance for  $G_{\alpha\beta\gamma}$ , found that it had to be proportional to the triangle diagram, and then found that this result had already been published by Schreier.<sup>55</sup>

So the evaluation of the VVA singular function was complete:

$$G_{\alpha\beta\gamma}(x, y) = \frac{KR'}{12\pi^6 u^2 v^2 w^2} \text{Tr}\{\gamma_\alpha \gamma \cdot x \gamma_\beta \gamma \cdot y \gamma_\gamma (\gamma \cdot x - \gamma \cdot y) \gamma_5\}. \quad (41)$$

When  $G_{\alpha\beta\gamma}$  is substituted into Eq. (24), the equal-time commutator regions give no contribution (as before), so as long as (say)  $\epsilon'$  is held fixed, the limits  $\epsilon \rightarrow 0$  and  $\epsilon'' \rightarrow 0$  can be taken without intruding on the short-distance region. Thus<sup>c</sup>

$$S = \frac{\pi^2}{12} \int d^3 y \int d^4 x \{ \tilde{Y}_0(y_0 = \epsilon') - \tilde{Y}_0(y_0 = -\epsilon') \} + O(\epsilon') \quad (42)$$

where  $\tilde{Y}_\gamma$  is the  $x, y \sim 0$  part of  $Y_\gamma$ :

$$\begin{aligned} \tilde{Y}_\gamma &= \epsilon^{\mu\nu\alpha\beta} x_\mu y_\nu \{ G_{\alpha\beta\gamma}(x, y) + G_{\alpha\gamma\beta}(x, y) \} \\ &= -\frac{4KR'}{3\pi^6 u^2 v^2 w^2} y_\gamma \{ x^2 y^2 - (x \cdot y)^2 \}. \end{aligned} \quad (43)$$

Do the  $x$ -integral

$$\int d^4 x \{ x^2 y^2 - (x \cdot y)^2 \} / (uw)^2 = -3\pi^2 i / 2 \quad (44)$$

<sup>b</sup>This has *nothing* to do with the properties of the vacuum state. As noted at the beginning of Sec. 3,  $|\text{vac}\rangle$  breaks scale and hence conformal invariance very strongly. This may be due to explicit symmetry breaking or to the symmetry being realised in the Nambu-Goldstone mode.

<sup>c</sup>These details, taken from a letter I wrote to Fritzsche and Gell-Mann at the time,<sup>56</sup> should have been part of ref. [10] of my paper<sup>18</sup> but it was never finished.

and then the  $y$ -integral

$$\int d^3y/v^2 = -\pi^2 i/|y_0| \quad (45)$$

to obtain the desired formula<sup>18</sup>

$$3S = KR'. \quad (46)$$

It relates the low-energy amplitude  $S$  to high-energy amplitudes  $R'$  and  $K$ .

Being anxious to avoid model dependence, I allowed for the possibility that the electromagnetic current is not a pure  $SU(3)$  octet,

$$4R' \leq 3R. \quad (47)$$

At that time, we did not know  $R$ ,  $R'$  or  $K$ . In particular, data showing scaling behavior for  $e^+ + e^- \rightarrow$  hadrons was not available until the 1974 London Conference, a year after asymptotic freedom. The value of  $S$  could well be exactly 0.5, so Adler's non-renormalization theorem<sup>25,57</sup> for  $S$  suggested a theory with three species of quark, but I could not see how to deal with colored electromagnetic currents or paraquark operators.

So when I received by return mail a letter from Gell-Mann proposing colored fractional quarks with color-neutral currents, it seemed to me that this clarified matters from the point of view of Adler's theorem, but I felt (for reasons discussed above) that having free quarks on the light cone was going too far. However, sometimes an oversimplification can lead to correct answers — in this case, QCD<sup>58</sup> and asymptotic freedom.<sup>32,33</sup>

Initially, the QCD proposal looked good as a model of constituent quarks, but not for current quarks and the  $\pi^0 \rightarrow \gamma\gamma$  analysis. Having found the electromagnetic trace anomaly, we knew already that  $\theta_\mu^\mu$  would have anomalous gluonic terms proportional to  $G_{\mu\nu}^a G^{a\mu\nu}$  which would break scale and conformal invariance at short distances.

What asymptotic freedom did was to turn QCD into a good theory of current quarks as well as constituent quarks. The breaking of scale invariance at short distances was minimal, being associated with operators which are not conserved exactly or partially. The analysis of  $\pi^0 \rightarrow \gamma\gamma$  can be still be carried through since all of the equations remain valid: they can be derived by using asymptotic freedom instead of asymptotic conformal invariance. The results for three colors are

$$R = 2, \quad R' = 1.5, \quad K = 1, \quad \text{and} \quad S = 0.5 \quad (48)$$

where the non-renormalization theorem for  $S$  is *not* used. All of this goes through *without* treating pions perturbatively.

The method of nested operator product expansions is now not needed for the  $\pi^0 \rightarrow \gamma\gamma$  derivation, but it is generally valid in renormalized field theory. As a result, coupling constant dependence of the form

$$3S = K(g)R'(g) \quad (49)$$

can be investigated.<sup>59</sup> This program has been extensively pursued by Andrei Kataev, Stan Brodsky and their collaborators.<sup>60</sup>

In retrospect, there came a time when abstracting physics had to give way to guessing the correct model. I remember that time well.

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